

Distributive Minimization Comprehensions and the Polynomial Hierarchy

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Abstract

A categorical point of view about minimization in subrecursive classes is presented by extending the concept of Symmetric Monoidal Comprehension to that of Distributive Minimization Comprehension. This is achieved by endowing the former with coproducts and a finality condition for coalgebras over the endofunctor sending X to $1 \oplus X$ to perform a safe minimization operator. By relying on the characterization given by Bellantoni, a tiered structure is presented from which one can obtain the levels of the Polytime Hierarchy as those classes of partial functions obtained after a certain number of minimizations.

Keywords: Safe Recursion, Safe Minimization, Distributive Monoidal Categories, Polytime Hierarchy.

1 Introduction

The safe interpretation of recursion was introduced by Bellantoni and Cook in [2] and can be used to substitute the bounding condition in the bounded recursion scheme

$$\begin{cases} f(u, 0) = g(u) \\ f(u, x + 1) = h(u, x, f(u, x)) \\ f(u, x) \leq j(u, x) \end{cases}$$

under which the subrecursive function classes are closed by a syntactical condition. The central idea of Bellantoni and Cook was to define two different kinds of variables (*normal* and *safe variables*) according to the use we make of them

in the process of computation. In [2] the class of polynomial time functions has been characterized in safety terms. In particular, the authors define a class of functions in the form $f(\overline{x}; \overline{y})$ where each input in f is called *normal* or *safe input*, normal inputs are in the left and separate them from safe by making use of a semicolon.

In its turn, the *ramified recursion* is a way to avoid *impredicativity* problems. In a ramified system the objects are defined using levels such that the definition of an object in level i depends only on levels below i . We will make use of some sets \mathbb{N}_k , the *levels of the natural numbers*, since they have a close relation with different function classes according to their complexity degree.

Following the previous ideas, Bellantoni gives in [1] a characterization in safety terms of the known as *Polytime Hierarchy* as that collection of classes \square_{i+1}^P :

- containing the initial functions
 - zero function
 - projections: $\pi_j^{m,p}(x_1, \dots, x_m; x_{m+1}, \dots, x_{m+p}) = x_j$ for $1 \leq j \leq m+p$
 - binary successors: $s^1(; m) = 2m$ and $s^2(; m) = 2m + 1$
 - predecessor: $p(; s^1(; 0)) = p(; s^2(; 0)) = 0$
 - conditional modulo: $C(; a, b, c) = \begin{cases} b & \text{if } a \bmod 2 = 0 \\ c & \text{otherwise} \end{cases}$
- closed under
 - safe composition:

$$f(x; a) = h(\overline{r}(\overline{x}); t(\overline{x}; \overline{a}))$$

- for $n = 1, 2$ predicative recursion on notation:

$$\begin{cases} f(0, \overline{x}; \overline{a}) = g(\overline{x}; \overline{a}) \\ f(s^n(y), \overline{x}; \overline{a}) = h_n(y, \overline{x}; \overline{a}, f(y, \overline{x}; \overline{a})) \quad \text{for } s^n(; 0) \neq 0 \end{cases}$$

- and obtained after i applications of safe minimization:¹

$$f(\overline{x}; \overline{a}) = \begin{cases} s^2(\mu b. h(\overline{x}; \overline{a}, b) \bmod 2 = 0) & \text{if there is such } b \\ 0 & \text{otherwise} \end{cases}$$

In [4] the author develops a categorical setting to characterize subrecursive hierarchies in categorical terms based on the safe and ramified interpretation of recursion referred to above. For that it is introduced the concept of *Symmetric Monoidal Comprehension*. From that construction it is proved that one can perform functions in a growing classification such as the *Grzegorzcyk Hierarchy*.

¹The safe minimization operator is total and does not entail a notion of partiality, this is explained in [1].

The aim of this paper is to extend the results of [4] giving a categorical setting to study safe minimization in the context of subrecursive classes closed under the operations of safe recursion and composition, and it is based on former works where that operator was not considered. In particular, the novelty of this work is to consider two subindices rather than the one considered in [4]. The first subindex, ranging in $\{0, 1\}$, relates to the normal and safe variables into the functions defined by safe recursion, the second one, belonging to the set $\{0, \dots, i - 1\}$, is devoted to count the number of safe minimizations computed to obtain a function in a certain class of the Polynomial Hierarchy.

It is known, at the same time, that initiality for algebras associated to endofunctors $F(-) = 1 \oplus -$ is used to perform recursion while finality for the same endofunctor allows to interpret minimization. Some conditions, essentially sums, have to be added to a monoidal category to perform that operator, the closing operation required to get the class of partial recursive functions. In this paper Symmetric Monoidal Comprehensions are endowed with more structure to obtain, by means of the safe recursion scheme, a condition of distributivity and, with the above-mentioned finality condition, safe minimization. This gives rise to the concept of *Distributive Minimization Comprehension*, the setting from which we represent partial subrecursive functions and the Polytime Hierarchy in particular.

Certain endofunctors M_p for $p \in \{0, \dots, i - 1\}$ allow to bound the number of times that we compute safe minimization in the finality diagram. After all this is well established, the *Freyd Cover* plays the role of representability in the context of partial functions (see [6]).

The article is structured as follows: section 2 introduces the basic concepts and that of Distributive Minimization Comprehension in particular by giving our main example, in section 3 it is proved that distributivity is a condition obtained from a *SRR* scheme while section 4 deals with safe minimization following the ideas introduced in [7], essentially finality for coalgebras over a certain functor. In section 5 we explain how to represent recursive functions in the free Distributive Minimization Comprehension and which are the important features satisfied by it. Finally, in section 6 some conclusions and lines for further development are given.

2 Basic structures

We begin by considering the categories $\Delta^{op}(\mathbf{i}, \mathbf{i})$ and $\Delta^{op}(\mathbf{2}, \mathbf{2})$, where Δ is the simplicial category, as the monoids of endofunctors in \mathbf{i} and $\mathbf{2}$. That is, the categories with objects the natural numbers lower than i and 2 and arrows $0 \rightarrow 1 \rightarrow \dots \rightarrow i - 1$ and $0 \rightarrow 1$ respectively.

Definition. Let be:

- the functors T and G in $\Delta^{op}(\mathbf{2}, \mathbf{2})$ such that $Tk = 1$ and $Gk = 0$ for $k = 0, 1$,

- for every $p, m \in \mathbf{i}$ the functors M_p in $\Delta^{op}(\mathbf{i}, \mathbf{i})$ such that

$$M_p(m) = \begin{cases} p+1 & \text{if } m = p \\ m & \text{if } m \neq p \end{cases}$$

- for all $k \in \mathbf{2}$ and $\epsilon : G \Rightarrow id$ and $\eta : id \Rightarrow T$ natural transformations such that

$$\epsilon(k) = \begin{cases} id_1 & \text{if } k \neq 1 \\ 0 \rightarrow 1 & \text{if } k = 1 \end{cases} \quad \eta(k) = \begin{cases} id_0 & \text{if } k \neq 0 \\ 0 \rightarrow 1 & \text{if } k = 0 \end{cases}$$

A category have the same certain bicategorical property than another category if the same commutative diagrams are satisfied for both of them, that is, if there exists a bifunctor between them. For the definition of *Distributive i-Minimization Comprehension* we consider certain properties that one category *inherits* from other. This is the basic categorical structure from which we will develop recursion for subrecursive (partial) function classes. We endow a categorical structure with initial diagrams and recursive operators.

In the following the indices range as indicated here: $p, q \in \mathbf{i}$, $k \in \mathbf{2}$, $n = 1, 2$ and $\alpha \in \mathbb{N}$.

Definition. A *Distributive i-Minimization Comprehension*, denoted in the sequel by $(\mathcal{C}, T^{\mathcal{C}}, G^{\mathcal{C}}, \eta^{\mathcal{C}}, \epsilon^{\mathcal{C}}, M_p^{\mathcal{C}})$

1. consists of:

- a symmetric monoidal category with coproducts \mathcal{C} ,²
- the functors $T^{\mathcal{C}}, G^{\mathcal{C}}, M_p^{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ preserving \otimes and \oplus on the nose³,
- natural transformations $\eta^{\mathcal{C}} : id \Rightarrow T^{\mathcal{C}}$ and $\epsilon^{\mathcal{C}} : G^{\mathcal{C}} \Rightarrow id$,
- bifunctors $\mathfrak{S}_{Rec} : \Delta^{op}(\mathbf{2}, \mathbf{2}) \rightarrow (\mathcal{C}, \mathcal{C})$ and $\mathfrak{S}_{Min} : \Delta^{op}(\mathbf{i}, \mathbf{i}) \rightarrow (\mathcal{C}, \mathcal{C})$ such that⁴

$$\mathfrak{S}_{Rec}(T) = T^{\mathcal{C}} \quad \mathfrak{S}_{Rec}(G) = G^{\mathcal{C}}$$

$$\mathfrak{S}_{Rec}(\eta) = \eta^{\mathcal{C}} \quad \mathfrak{S}_{Rec}(\epsilon) = \epsilon^{\mathcal{C}} \quad \mathfrak{S}_{Min}(M_p) = M_p^{\mathcal{C}}$$

2. containing an object $N_{0,p}$ and three arrows $0_{0,p}$, $s_{0,p}^1$ and $s_{0,p}^2$ with *initial diagrams*

$$\top \xrightarrow{0_{0,R}} N_{0,p} \xrightarrow{s_{0,R}^1} N_{0,p} \quad \top \xrightarrow{0_{0,R}} N_{0,p} \xrightarrow{s_{0,R}^2} N_{0,p}$$

for binary numbers. We define recursively the objects $N_{1,p}$ by the rule

$$N_{1,p} = G^{\mathcal{C}} N_{0,p}$$

²We denote the elements of that structure by $\oplus, in_r, in_l, \otimes, \top, l$ and express the objects *modulo associativity* and *symmetry* in the sequel.

³Preservation *on the nose* means for us equations such as $T^{\mathcal{C}}(A \otimes B) = T^{\mathcal{C}}A \otimes T^{\mathcal{C}}B$, $T^{\mathcal{C}}(f \otimes B) = T^{\mathcal{C}}f \otimes T^{\mathcal{C}}B$, $T^{\mathcal{C}}\top = \top$ etc. and same for $G^{\mathcal{C}}$ and $M_p^{\mathcal{C}}$.

⁴For both \mathfrak{S} to exist we are looking at $\Delta^{op}(\mathbf{i}, \mathbf{i})$ and $\Delta^{op}(\mathbf{2}, \mathbf{2})$ as bicategories with a unique 0-cells \mathbf{i} and $\mathbf{2}$ respectively.

and morphisms $0_{1,p}$, $s_{1,p}^1$ and $s_{1,p}^2$ defined by $0_{1,p} = G^{\mathcal{C}}(0_{0,p})$ and $s_{1,p}^1 = G^{\mathcal{C}}(s_{0,p}^1) = G^{\mathcal{C}}(s_{0,p}^2)$ giving initial diagrams for $N_{1,p}$. We also have in \mathcal{C}

$$\begin{aligned} T^{\mathcal{C}} N_{0,p} &= \top & T^{\mathcal{C}} N_{1,p} &= N_{1,p} \\ G^{\mathcal{C}} N_{1,p} &= N_{1,p} \end{aligned}$$

As well as

$$M_p^{\mathcal{C}} N_{k,q} = \begin{cases} \top & \text{if } p = q = 0 \\ N_{k,p-1} & \text{if } p = q \neq 0 \\ N_{k,q} & \text{otherwise} \end{cases}$$

3. closed under

- *flat recursion*:
for all morphisms

$$g : X \longrightarrow Y \text{ and } h : N_{0,p} \otimes X \longrightarrow Y$$

where X and Y are in the form $N_{0,p}^{\alpha}$ there exist a unique

$$FR(g, h) : N_{0,p} \otimes X \longrightarrow Y$$

in \mathcal{C} such that the following diagrams commute

$$\begin{array}{ccccc} \top \otimes X & \xrightarrow{0_{0,p} \otimes X} & N_{0,p} \otimes X & \xleftarrow{s_{0,p}^n \otimes X} & N_{0,p} \otimes X \\ & \searrow g \circ l & \downarrow FR(g, h) & \swarrow h & \\ & & Y & & \end{array}$$

- *safe ramified recursion*:
for all morphisms

$$g : X \longrightarrow Y \text{ and } h : Y \longrightarrow Y$$

where Y belongs to the fiber of $T^{\mathcal{C}}$ over \top there exist a unique

$$SRR(g, h) : N_{1,p} \otimes X \longrightarrow Y$$

in \mathcal{C} such that the following diagram commutes

$$\begin{array}{ccccc} \top \otimes X & \xrightarrow{0_{1,p} \otimes X} & N_{1,p} \otimes X & \xrightarrow{s_{1,p}^n \otimes X} & N_{1,p} \otimes X \\ \downarrow l & & \downarrow SRR(g, h) & & \downarrow SRR(g, h) \\ X & \xrightarrow{g} & Y & \xrightarrow{h} & Y \end{array}$$

4. and such that every arrow $0_{k,p} \oplus s_{k,p}^n$ is an isomorphism such that the pair

$$(N_{k,p}, (0_{k,p} \oplus s_{k,p}^n)^{-1})$$

is a *bounded terminal coalgebra* for the endofunctor $1 \oplus -$ in \mathcal{C} in the following sense:

for arrows $h_1, h_2 : A \rightarrow 1 \oplus A$ and $f : A \rightarrow N_{k,p-1}$ there is a unique $\mu f : A \rightarrow N_{k,p}$ such that the following diagram commute

$$\begin{array}{ccc} A & \xrightarrow{h_n} & 1 \oplus A \\ \mu f \downarrow & & \downarrow 1 \oplus \mu f \\ N_{k,p} & \xrightarrow{(0_{k,p} \oplus s_{k,p}^n)^{-1}} & 1 \oplus N_{k,p} \end{array}$$

where $N_{k,p}$ belongs to the fiber of $M_{i-1}^C \dots M_0^C$ over \top .

Flat recursion schemes are actually coproduct diagrams from which, by applying G , we obtain flat recursion also for $N_{1,p}$, they give the initial diagrams appropriate properties such as the injectivity of *successor functions* s^n .

Moreover, flat recursion schemes allow to define the predecessor function p given in the Introduction as $FR(0, id)$ as well as the conditional modulo function C with the help of the *conditional on test for zero* function Z :

$$\begin{array}{ccccc} \top \otimes N_{0,p} & \xrightarrow{0_{1,p} \otimes N_{0,p}} & N_{1,p} \otimes N_{0,p} & \xleftarrow{s_{1,p}^n \otimes N_{0,p}} & N_{1,p} \otimes N_{0,p} \\ & \searrow \pi_0 \pi_1 \circ l & \downarrow Z & \swarrow \pi_1 \pi_1 & \\ & & N_{0,p} & & \end{array}$$

Then, the composition of Z with $\text{mod } 2(a) = a \text{ mod } 2$ defined by

$$\begin{array}{ccccc} \top & \xrightarrow{0_{1,p}} & N_{1,p} & \xrightarrow{s_{1,p}^n} & N_{1,p} \\ & \searrow 0_{0,p} \circ l & \downarrow \text{mod } 2 & & \downarrow \text{mod } 2 \\ & & N_{0,p} & \xrightarrow{1-} & N_{0,p} \end{array}$$

gives the conditional modulo function C where $1-$ is the function $a \mapsto 1-a$ and $-$ is the non-negative substraction.

Remark. To define Z we have made use of projections which are not at our disposal unless we are in the context of a cartesian category. But this is precisely the case for the *free Distributive i-Minimization Comprehension* defined in section 5 (see Theorem 5).

Condition 4. gives a *safe minimization operator* as explained in section 3 for the following example. The bounding condition for that operator over the codomain of μf ensures that we do not compute safe minimization more than i times.

Example. Our example of Distributive i-Minimization Comprehension consists of defining that structure for a presheaf over the category of sets and partial functions Set_P .

Consider the category $Set_P^{2 \times i}$. Its objects are squares formed by chains of sets indexed by $2 \times i$:

$$\begin{array}{ccccccc} X_{0,0} & \longrightarrow & X_{0,1} & \longrightarrow & \cdots & \longrightarrow & X_{0,(i-1)} \\ \downarrow & & \downarrow & & & & \downarrow \\ X_{1,0} & \longrightarrow & X_{1,1} & \longrightarrow & \cdots & \longrightarrow & X_{1,(i-1)} \end{array}$$

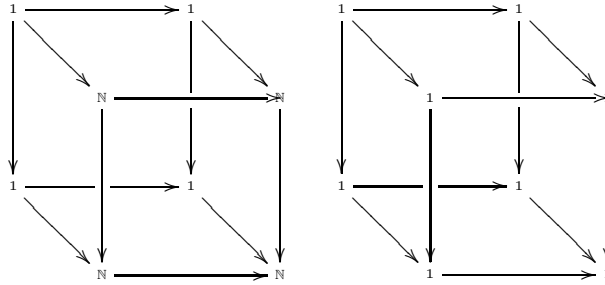
and its arrows cubes built out of them.

- $Set_P^{2 \times i}$ is a symmetric monoidal category with coproducts,
- it has as terminal object chains

$$\begin{array}{ccccccc} 1 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & 1 \\ \downarrow & & \downarrow & & & & \downarrow \\ 1 & \longrightarrow & 1 & \longrightarrow & \cdots & \longrightarrow & 1 \end{array}$$

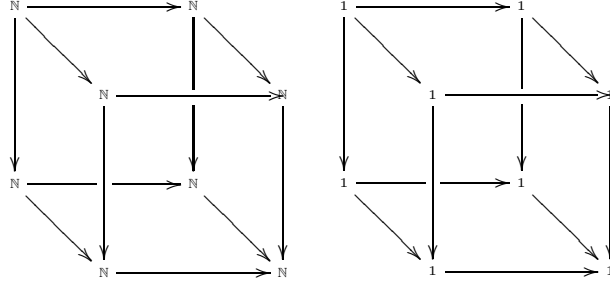
denoted by $1^{2 \times i}$ where 1 is any set with a single object and

- for $p \in i$:
 - $0_{k,p}$ give rise to $p - 1$ cubes as in the left and $i - p + 1$ cubes such as the one at right:



filled with zero and identity arrows

- $s_{k,p}^n$ give rise to $p - 1$ cubes as in the left and $i - p + 1$ cubes such as the one at right:



with binary successors and identity arrows,

- Fixing a single object X there are some special objects in the form

$$\begin{array}{ccccccc}
 X & \longrightarrow & X & \longrightarrow & \cdots & \longrightarrow & 1 & \longrightarrow & 1 \\
 \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 X & \longrightarrow & X & \longrightarrow & \cdots & \longrightarrow & 1 & \longrightarrow & 1
 \end{array}$$

and denoted by $X^{p,q}$ where the chain above is formed by p objects X and $i - p$ objects 1 and the chain below is formed by q objects X and $i - q$ objects 1 . We call these objects the *levels of X* .

- We define preserving endofunctors T^S and G^S acting over the columns of $X^{p,q}$ as:

$$\begin{array}{cc}
 T^S \left[\begin{array}{c} X \\ \downarrow \\ 1 \end{array} \right] = \begin{array}{c} 1 \\ \downarrow \\ 1 \end{array} & T^S \left[\begin{array}{c} X \\ \downarrow \\ X \end{array} \right] = \begin{array}{c} X \\ \downarrow \\ X \end{array} \\
 G^S \left[\begin{array}{c} X \\ \downarrow \\ 1 \end{array} \right] = \begin{array}{c} X \\ \downarrow \\ X \end{array} & G^S \left[\begin{array}{c} X \\ \downarrow \\ X \end{array} \right] = \begin{array}{c} X \\ \downarrow \\ X \end{array}
 \end{array}$$

and in general over every arrow

$$\begin{array}{c}
 X_{0,p} \\
 \downarrow \\
 X_{1,p}
 \end{array}$$

as:

$$T^S \begin{bmatrix} X_{0,p} \\ \downarrow \\ X_{1,p} \end{bmatrix} = \begin{array}{c} 1 \\ \downarrow \\ X_{1,p} \end{array} \quad G^S \begin{bmatrix} X_{0,p} \\ \downarrow \\ X_{1,p} \end{bmatrix} = \begin{array}{c} X_{1,p} \\ \downarrow \\ X_{1,p} \end{array}$$

It is obvious that they preserve all tensor and coproducts.

- While for endofunctors M_p^S we have for the rows

$$X^{(p)} = X \longrightarrow \dots \xrightarrow{p} X \longrightarrow 1 \longrightarrow \dots \longrightarrow 1$$

the following table:

	$X^{(0)}$	$X^{(1)}$...	$X^{(i-3)}$	$X^{(i-2)}$	$X^{(i-1)}$
M_0^S	$1^{2 \times i}$	$X^{(1)}$...	$X^{(i-3)}$	$X^{(i-2)}$	$X^{(i-1)}$
M_1^S	$X^{(0)}$	$X^{(0)}$...	$X^{(i-3)}$	$X^{(i-2)}$	$X^{(i-1)}$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
M_{i-1}^S	$X^{(0)}$	X^1	...	$X^{(i-3)}$	$X^{(i-3)}$	$X^{(i-1)}$

while in general M_p^S acts over every chain

$$X_{0,0} \xrightarrow{h_0} X_{0,1} \xrightarrow{h_1} \dots \xrightarrow{h_{i-2}} X_{0,(i-1)}$$

as:

$$X_{0,0} \xrightarrow{h_0} \dots \xrightarrow{h_{i-p}} X_{0,i-p+1} \xrightarrow{id} X_{0,i-p+1} \xrightarrow{t} X_{0,i-p+3} \xrightarrow{h_{i-p+3}} \dots \xrightarrow{h_{i-2}} X_{0,(i-1)}$$

where $t = h_{i-p+2} \circ h_{i-p+1}$, that is, it repeats the $(i-p+1)$ -term. It is obvious that it preserves all tensor and coproducts.

- We define bifunctors $\mathfrak{S}_{Rec} : \Delta^{op}(\mathbf{2}, \mathbf{2}) \rightarrow (Set_P^{2 \times i}, Set_P^{2 \times i})$ and $\mathfrak{S}_{Min} : \Delta^{op}(\mathbf{i}, \mathbf{i}) \rightarrow (Set_P^{2 \times i}, Set_P^{2 \times i})$ sending T, G, η, ϵ and M_p to the respective endofunctors and natural transformations for $Set_P^{2 \times i}$.
- The category of coalgebras for the endofunctor sending an object X to $1 \oplus X$ in $Set_P^{2 \times i}$ is endowed with a number of isomorphic terminal objects (see section 4).

3 Distributivity

In this section we prove that, as a consequence of the previous definition, we are actually endowing \mathcal{C} with a structure of distributive monoidal category where the distributive arrows d are uniquely defined by an application of safe ramified recursion:

$$\begin{array}{ccccc}
\top \otimes (X \oplus Y) & \xrightarrow{0 \otimes (X \oplus Y)} & N_{1,p} \otimes (X \oplus Y) & \xrightarrow{s \otimes (X \oplus Y)} & N_{1,p} \otimes (X \oplus Y) \\
\downarrow l & & \downarrow d_{N_{1,p}, X, Y} & & \downarrow d_{N_{1,p}, X, Y} \\
X \oplus Y & \xrightarrow{(0 \otimes X)l_X^{-1} \oplus (0 \otimes Y)l_Y^{-1}} & (N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y) & \xrightarrow{(s \otimes X) \oplus (s \otimes Y)} & (N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y)
\end{array}$$

The arrows d are actually isomorphisms because of the identities contained in the following result where the inclusions appearing are both isomorphisms. We omit subscripts in the sequel.

Proposition. *For every m and arrows $x : \top \rightarrow X, y : \top \rightarrow Y$ the following diagrams commute:*

$$\begin{array}{ccc}
\top \otimes \top & \xrightarrow{\hat{m} \otimes (in_l \circ x)} & N_{1,p} \otimes (X \oplus Y) \\
\hat{m} \otimes x \downarrow & & \downarrow d \\
N_{1,p} \otimes X & \xrightarrow{in_l} & (N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y)
\end{array}
\quad
\begin{array}{ccc}
\top \otimes \top & \xrightarrow{\hat{m} \otimes (in_r \circ y)} & N_{1,p} \otimes (X \oplus Y) \\
\hat{m} \otimes y \downarrow & & \downarrow d \\
N_{1,p} \otimes Y & \xrightarrow{in_r} & (N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y)
\end{array}$$

where we denote \hat{m} for the arrows $(s_{1,p})^m 0_{1,p} : \top \rightarrow N_{1,p}$.⁵

Proof. We proceed by induction over m .

- For $m = 1$ the following diagram is a composition of commuting diagrams:

$$\begin{array}{ccc}
\top \otimes \top & \xrightarrow{\hat{1} \otimes (in_l \circ x)} & N_{1,p} \otimes (X \oplus Y) \\
\downarrow \hat{1} \otimes x & & \downarrow d \\
\top \otimes X & \xrightarrow{\hat{1} \otimes in_l} & X \\
\downarrow \hat{1} \otimes X & & \downarrow g \\
N_{1,p} \otimes X & \xrightarrow{in_l} & (N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y)
\end{array}$$

(α) (β) (γ) (δ)

where $g = (s_0 \otimes X) \oplus (s_0 \otimes Y) \circ (l_X^{-1} \oplus l_Y^{-1}) \circ in_l$. Diagrams α , β and γ commute by direct inspection, δ commutes because it can be expressed in the form

⁵We do not distinguish between s^1 and s^2 and write just s since it does not make any difference.

$$\begin{array}{ccccc}
\top \otimes X & \xrightarrow{i \otimes X} & N_{1,p} \otimes X & \xrightarrow{s \otimes in_l} & N_{1,p} \otimes (X \oplus Y) \\
\downarrow \iota_X & & \downarrow in_l & & \downarrow d \\
X & \xrightarrow{g} & (N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y) & \xrightarrow{f} & (N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y)
\end{array}$$

(ϵ) (η)

where $f = (s \otimes X) \oplus (s \otimes Y)$. In it ϵ commutes trivially and η commutes because the following diagram commutes:

$$\begin{array}{ccc}
N_{1,p} \otimes X & \xrightarrow{s \otimes in_l} & N_{1,p} \otimes (X \oplus Y) \\
\downarrow N_{1,p} \otimes in_l & & \downarrow d \\
N_{1,p} \otimes (X \oplus Y) & \xrightarrow{s \otimes (X \oplus Y)} & N_{1,p} \otimes (X \oplus Y) \\
\downarrow d & & \downarrow d \\
(N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y) & \xrightarrow{f} & (N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y)
\end{array}$$

in_l (curved arrow from $N_{1,p} \otimes X$ to $(N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y)$)

- If we suppose true our result for \hat{m} the following diagrams commute

$$\begin{array}{ccc}
\top \otimes \top & \xrightarrow{(\widehat{m+1}) \otimes (in_l \circ x)} & N_{1,p} \otimes (X \oplus Y) \\
\downarrow (\widehat{m+1}) \otimes x & & \downarrow d \\
N_{1,p} \otimes X & \xrightarrow{in_l} & (N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y)
\end{array}$$

$$\begin{array}{ccc}
\top \otimes \top & \xrightarrow{(\widehat{m+1}) \otimes (in_r \circ y)} & N_{1,p} \otimes (X \oplus Y) \\
\downarrow (\widehat{m+1}) \otimes y & & \downarrow d \\
N_{1,p} \otimes Y & \xrightarrow{in_r} & (N_{1,p} \otimes X) \oplus (N_{1,p} \otimes Y)
\end{array}$$

by composition.

□

This result, that justifies the word *distributive* in the name of the structure, can be extended to whatever power of the levels of natural numbers. That is, the following diagrams also commute

$$\begin{array}{ccc}
\top \otimes \top & \xrightarrow{\langle \widehat{m_1}, \dots, \widehat{m_\alpha} \rangle \otimes (in_l \circ x)} & N_{p,1}^\alpha \otimes (X \oplus Y) \\
\downarrow \langle \widehat{m_1}, \dots, \widehat{m_\alpha} \rangle \otimes x & & \downarrow d \\
N_{1,p}^\alpha \otimes X & \xrightarrow{in_l} & (N_{1,p}^\alpha \otimes X) \oplus (N_{1,p}^\alpha \otimes Y)
\end{array}$$

$$\begin{array}{ccc}
\top \otimes \top & \xrightarrow{\langle \widehat{m}_1, \dots, \widehat{m}_\alpha \rangle \otimes (in_r \circ y)} & N_{1,p}^\alpha \otimes (X \oplus Y) \\
\downarrow \langle \widehat{m}_1, \dots, \widehat{m}_\alpha \rangle \otimes y & & \downarrow d \\
N_{1,p}^\alpha \otimes Y & \xrightarrow{in_r} & (N_{1,p}^\alpha \otimes X) \oplus (N_{1,p}^\alpha \otimes Y)
\end{array}$$

where we consider arrows

$$\langle \widehat{m}_1, \dots, \widehat{m}_\alpha \rangle : \top \rightarrow N_{1,p}^\alpha$$

We have in fact the following relations into a Distributive i-Minimization Comprehension $(\mathcal{C}, T, G, \eta, \epsilon, M_p)$, ensuring coherence in the complexity growing structure:

$$Td_{N_{0,p},X,Y} = d_{\top, TX, TY} \quad Gd_{N_{0,p},X,Y} = d_{N_{1,p}, GX, GY}$$

$$M_p d_{N_{k,p},X,Y} = d_{N_{k,p-1}, M_p X, M_p Y}$$

for $k \in \mathbf{2}$, $p \in \mathbf{i} \setminus \{0\}$ and every $X, Y \in \mathcal{C}$.

4 Coalgebras and partiality

In this section we treat partiality in a Distributive i-Minimization Comprehension. We start from the well known idea that the initial algebra $(\mathbb{N}, 1 \oplus \mathbb{N} \xrightarrow{0 \oplus s} \mathbb{N})$ of the endofunctor $1 \oplus -$ over the category *Set* turns out to be a strong natural numbers object (*nno* in the sequel) where the uniqueness condition included into it has its counterpart into the uniqueness of the *nno*: the equations obtained through a diagram

$$\begin{array}{ccc}
1 \oplus \mathbb{N} & \xrightarrow{0 \oplus s} & \mathbb{N} \\
1 \oplus h \downarrow & & \downarrow h \\
1 \oplus A & \longrightarrow & A
\end{array}$$

for every other algebra $(A, 1 \oplus A \rightarrow A)$ are equivalent to those obtained in a *nno* diagram.

It is precisely from duality that we obtain partiality for our recursive arrows: the terminal coalgebra for $1 \oplus -$ over *Set_P* gives a categorical intuition of minimization (see [7]). Let us denote $F : \text{Set}_P \rightarrow \text{Set}_P$ the endofunctor such that $FA = 1 \oplus A$ and its terminal coalgebra $(\mathbb{N}, \mathbb{N} \xrightarrow{\alpha} 1 \oplus \mathbb{N})$ where α turns out to be the isomorphism $(0 \oplus s)^{-1}$.

We spell out the details involved in this construction for the case of *Set_P*. For arrows h_1, h_2 and $f : \mathbb{N}^{\alpha_1} \otimes \mathbb{N}^{\alpha_0} \rightarrow \mathbb{N}$ the usual coalgebra diagram gives a unique $\mu f : \mathbb{N}^{\alpha_1} \otimes \mathbb{N}^{\alpha_0} \rightarrow \mathbb{N}$ such that the following diagrams commute

$$\begin{array}{ccc}
\mathbb{N}^{\alpha_1} \otimes \mathbb{N}^{\alpha_0} & \xrightarrow{h_n} & 1 \oplus (\mathbb{N}^{\alpha_1} \otimes \mathbb{N}^{\alpha_0}) \\
\mu f \downarrow & & \downarrow 1 \oplus \mu f \\
\mathbb{N} & \xrightarrow{(0 \oplus s^n)^{-1}} & 1 \oplus \mathbb{N}
\end{array}$$

The arrow μf is in this way defined by an analogous of Kleene's minimization from the partial function f (see [7]).

For our example 2 of Distributive i-Minimization Comprehension $Set_P^{2 \times i}$ described above we can investigate which is the form of the objects in the category of coalgebras for the endofunctor F analogous to the previous one.

Let $F^{2 \times i} : Set_P^{2 \times i} \rightarrow Set_P^{2 \times i}$ be such that

$$\begin{aligned}
F^{2 \times i} \left[\begin{array}{ccccccc} X_{0,0} & \longrightarrow & X_{0,1} & \longrightarrow & \cdots & \longrightarrow & X_{0,(i-1)} \\ \downarrow & & \downarrow & & & & \downarrow \\ X_{1,0} & \longrightarrow & X_{1,1} & \longrightarrow & \cdots & \longrightarrow & X_{1,(i-1)} \end{array} \right] = \\
= \begin{array}{ccccccc} 1 \oplus X_{0,0} & \longrightarrow & 1 \oplus X_{0,1} & \longrightarrow & \cdots & \longrightarrow & 1 \oplus X_{0,(i-1)} \\ \downarrow & & \downarrow & & & & \downarrow \\ 1 \oplus X_{1,0} & \longrightarrow & 1 \oplus X_{1,1} & \longrightarrow & \cdots & \longrightarrow & 1 \oplus X_{1,(i-1)} \end{array}
\end{aligned}$$

then the category $CoAlg F^{2 \times i}$ has:

- as objects pairs (X, α) where X is of the form

$$\begin{array}{ccccccc}
X_{0,0} & \longrightarrow & X_{0,1} & \longrightarrow & \cdots & \longrightarrow & X_{0,(i-1)} \\
\downarrow & & \downarrow & & & & \downarrow \\
X_{1,0} & \longrightarrow & X_{1,1} & \longrightarrow & \cdots & \longrightarrow & X_{1,(i-1)}
\end{array}$$

and $\alpha = (\alpha_{0,0}, \dots, \alpha_{0,i-1}, \alpha_{1,0}, \dots, \alpha_{1,i-1})$ with $\alpha_{k,p} : X_{k,p} \rightarrow 1 \oplus X_{k,p}$ for $k \in \mathbf{2}$ and $p \in \mathbf{i}$;

- as arrows $(X, \alpha) \xrightarrow{f} (Y, \beta)$ such that $\beta \circ f = (F^{2 \times i} f) \circ \alpha$.

Since $Set_P^{2 \times i}$ is an indexed category over Set_P and $CoAlg F$ has as terminal object $(\mathbb{N}, (0 \oplus s)^{-1})$ the category $CoAlg F^{2 \times i}$ has isomorphic terminal objects in the form of pairs formed by

- objects in the form

$$\begin{array}{ccccccc}
\mathbb{N}_{0,0} & \longrightarrow & \cdots & \longrightarrow & \mathbb{N}_{0,p} & \longrightarrow & 1 \longrightarrow \cdots \longrightarrow 1 \\
\downarrow & & & & \downarrow & & \downarrow \\
\mathbb{N}_{1,0} & \longrightarrow & \cdots & \longrightarrow & \mathbb{N}_{1,p} & \longrightarrow & 1 \longrightarrow \cdots \longrightarrow 1
\end{array}$$

which by definition of Distributive i-Minimization Comprehension belong to the fiber of $M_{i-1}^S \dots M_0^S$ over

$$\begin{array}{ccccccc} 1 & \longrightarrow & 1 & \longrightarrow & \dots & \longrightarrow & 1 \\ \downarrow & & \downarrow & & & & \downarrow \\ 1 & \longrightarrow & 1 & \longrightarrow & \dots & \longrightarrow & 1 \end{array}$$

- and cubes given by $2i$ arrows in the form $(0_{k,p} \oplus s_{k,p}^n)^{-1} : \mathbb{N}_{k,p} \rightarrow 1 \oplus \mathbb{N}_{k,p}$ for $k \in \mathbf{2}$ and $p \in \mathbf{i}$.

That is, $F^{2 \times i}$ can be endowed with a bounded terminal coalgebra. We have in this way a safe minimization operator, which is total according to [7], and is applied i -times as maximum in a Distributive i-Minimization Comprehension of partial functions.

5 The free Distributive i-Minimization Comprehension

By endowing the *initial symmetric monoidal category with coproducts* with all initial diagrams, the required recursion schemes and the terminal condition for coalgebras to obtain the minimization operator, we construct the *free Distributive i-Minimization Comprehension* which we denote \mathcal{DM}^i . The objects in \mathcal{DM}^i are of the form $\bigoplus_{k \in \mathbf{2}, p \in \mathbf{i}} \bigotimes N_{k,p}$, that is, coproducts of finite tensor products of the objects $N_{k,p}$ defined above. Moreover, it can be proved in this case that the tensor turns out to be a cartesian product. We have in this sense the following:

Theorem. \mathcal{DM}^i is a Distributive category.

Proof. See [4] together with Proposition 3. □

This result allows us to obtain the projection functions belonging to the Polynomial Hierarchy as defined in the Introduction.

It is precisely $\text{Set}_P^{2 \times i}$ from our example 2 that particular Distributive i-Minimization Comprehension in which we can represent the functions belonging to the i -level of the Polynomial Hierarchy \square_{i+1}^P . As in previous studies ([4, 6] for example) the image of \mathcal{DM}^i in $\text{Set}_P^{2 \times i}$ through the Freyd Cover will turn out to be exactly \square_{i+1}^P .

Definition. The *standard model of formal morphisms* is the functor Γ_i given by the diagram

$$\mathcal{DM}^i \xrightarrow{\Gamma_i} \text{Set}_P^{2 \times i}$$

as an \mathbf{i} -indexed version of the *Freyd Cover* for the functor $\Gamma : \mathcal{DM}^i \rightarrow \text{Set}_P$ defined by $\Gamma X = \mathcal{DM}^i(\top, X)$ and $\Gamma f = f \circ -$.⁶

⁶This is a special case of the *global sections functor*.

The syntactical structure here described is connected with the semantics of numerical functions in the sense that every arrow $\top \rightarrow N_{k,p}$ in \mathcal{DM}^i has the form $(s_{k,p}^n)^m 0_{k,p}$ for some $m \in \mathbb{N}$. In connection with this we have the following result.

Proposition. $\mathbb{N}_{k,p} = \{std_{k,p}m/m \in \mathbb{N}\}$ where $std_{k,p} : \mathbb{N} \rightarrow \mathbb{N}_{k,p}$ are defined by the schemes

$$\begin{cases} std_{k,p}0 = 0_{k,p} \\ std_{k,p}(s^n m) = s_{k,p}(std_{k,p}m) \end{cases}$$

with $k \in \mathbf{2}$ and $p \in \mathbf{i}$.

Corollary. $\Gamma N_{k,p} = \mathbb{N}_{k,p}$ for $k \in \mathbf{2}$ and $p \in \mathbf{i}$.

This Proposition and its Corollary indicate that the sets generated by the functor Γ applied to the levels of the natural numbers in \mathcal{DM}^i behave as the natural numbers themselves.

Safe composition, as defined in the Introduction, has a representation in a Distributive Minimization Comprehension by means of diagrams associated to the natural transformation η . For an arrow

$$f : N_{1,p}^\alpha \oplus N_{0,q}^\beta \longrightarrow N_{1,r}^\gamma$$

in \mathcal{DM}^i we have a commutative diagram in the form:

$$\begin{array}{ccc} N_{1,p}^\alpha \oplus N_{0,q}^\beta & \xrightarrow{f} & N_{1,r}^\gamma \\ \eta(N_{1,p}^\alpha \oplus N_{0,q}^\beta) \downarrow & & \downarrow \eta N_{1,r}^\gamma \\ T(N_{1,p}^\alpha \oplus N_{0,q}^\beta) & \xrightarrow{Tf} & T N_{1,r}^\gamma \end{array}$$

This grabs the formulation of safe composition given in the Introduction because we obtain an expression for morphisms in \mathcal{DM}^i with a normal output in terms of other morphisms whose safe inputs do not have any effect over normal outputs.

6 Conclusions and future work

It has been introduced the concept of Distributive i-Minimization Comprehension to extend the understanding of partiality in subrecursive functions, that idea has been addressed in [3] and in the context of recursion over arbitrary structures.

Some of the features of Symmetric Monoidal Comprehensions are inherited by this new categorical setting, essentially what is related with the free example. The main novelty of this structure is the double indexing of the objects relating the two safe operators involved in the construction of the Polynomial Hierarchy: *safe recursion* and *safe minimization*.

The consequence of adding coproducts is a distributive condition which is satisfied as an application of safe recursive schemes. On the other hand the terminal diagrams allow to develop minimization and, bounding the number of these operations that can be computed in every level, we obtain arrows whose representation in a certain category of sets are functions in the Polynomial Hierarchy.

There are several lines in which this work could be extended:

- look for representations of other subrecursive hierarchies of functions that could be characterized by modifying the concept of Symmetric Monoidal Comprehension or
- considering for Distributive i-Minimization Comprehensions, as done in [4] for Symmetric Monoidal Comprehensions, a modal interpretation of the many-sorted interpretation of recursion introduced primarily in [5].

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